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On the Stability and Hyperstability of a p-Radical Functional Equation Related to Jensen Mappings in 2-Banach Spaces

Muaadh Almahalebi, Sadeq Al-Ali

Abstract— The aim of this paper is to introduce and solve the following p-radical functional equation,

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=2f(x).$$

where f is a mapping from $\mathbb{R}$ into a vector space X and $p \geq 3$ is an odd natural number. Using an analogue version of the fixed point theorem in 2-Banach spaces, we establish some hyperstability results for the considered equation. Also, we study the hyperstability for the inhomogeneous p-radical functional equation related to Jensen mappings,

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=2f(x)+G(x,y).$$

Index Terms— stability , hyperstability, 2-Banach spaces, radical functional equations.

1 Introduction

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$ and $\mathbb{R}_+ = [0,\infty)$ the set of nonnegative real numbers. By $\mathbb{N}_m$, $m \in \mathbb{N}$, we will denote the set of all natural numbers greater than or equal to m.

The notion of linear 2-normed spaces was introduced by S. Gähler [22],[23] in the middle of 1960s. We need to recall some basic facts concerning 2-normed spaces and some preliminary results.

Definition 1.1 *Let X be a real linear space with $dimX > 1$ and $\|.,.\|: X \times X \to [0,\infty)$ be a function satisfying the following properties:*

1. $\| x,y \| = 0$ if and only if x and y are linearly dependent,

2. $\| x,y \| = \| y,x \|$,

3. $\| \lambda x,y \| = |\lambda| \, \| x,y \|$,

4. $\| x,y+z \| \leq \| x,y \| + \| x,z \|$,

for all $x,y,z \in X$ and $\lambda \in \mathbb{R}$. Then the function $\|.,.\|$ is called a 2-norm on X and the pair $(X,\|.,.\|)$ is called a linear 2-normed space. Sometimes the condition (4) called the triangle inequality.

Example 1.2 *For $x=(x_1,x_2)$, $y=(y_1,y_2) \in X = \mathbb{R}^2$, the Euclidean 2-norm $\| x,y \|_{\mathbb{R}^2}$ is defined by $\| x,y \|_{\mathbb{R}^2} = |x_1y_2 - x_2y_1|$.*

Lemma 1.3 *Let $(X,\|.,.\|)$ be a 2-normed space. If $x \in X$ and $\| x,y \| = 0$, for all $y \in X$, then $x=0$.*

Definition 1.4 *A sequence $\{x_k\}$ in a 2-normed space X is called a convergent sequence if there is an $x \in X$ such that*

$$\lim_{k\to\infty} \| x_k - x, y \| = 0,$$

for all $y \in X$. If $\{x_k\}$ converges to x, write $x_k \to x$ with $k \to \infty$ and call x the limit of $\{x_k\}$. In this case, we also write $lim_{k\to\infty} x_k = x$.

Definition 1.5 *A sequence $\{x_k\}$ in a 2-normed space X is said to be a Cauchy sequence with respect to the 2-norm if*

$$\lim_{k,l\to\infty} \| x_k - x_l, y \| = 0,$$

for all $y \in X$. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be complete with respect to the 2-norm. Any complete 2-normed space is said to be a 2-Banach space.

Now, we state the following results as lemma (See [27] for the details).

Lemma 1.6 *Let X be a 2-normed space. Then,*

1. *$|\, \| x,z \| - \| y,z \| \,| \leq \| x-y, z \|$ for all $x,y,z \in X$,*
2. *if $\| x,z \| = 0$ for all $z \in X$, then $x = 0$,*
3. *for a convergent sequence x_n in X,*
$$\lim_{n\to\infty} \| x_n, z \| = \left\| \lim_{n\to\infty} x_n, z \right\|$$

for all $z \in X$.

The concept of stability for a functional equation arises when defining, in some way, the class of approximate solutions of the given functional equation, one can ask whether each mapping from this class can be somehow approximated by an exact solution of the considered equation. Namely, when one replaces a functional equation by an inequality which acts as a perturbation of the considered equation. The first stability problem of functional equation was raised by S. M. Ulam [31] in 1940. This included the following question concerning the stability of group homomorphisms.

Let $(G_1,*_1)$ be a group and let $(G_2,*_2)$ be a metric group with a metric $d(.,.)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h: G_1 \to G_2$ satisfies the inequality

$$d(h(x *_1 y), h(x) *_2 h(y)) < \delta$$

for all $x,y \in G_1$, then there exists a homomorphism $H: G_1 \to G_2$ with $d(h(x),H(x)) < \varepsilon$ for all $x \in G_1$?

If the answer is affirmative, we say that the equation of homomorphism $h(x *_1 y) = h(x) *_2 H(y)$ is stable. Since then, this question has attracted the attention of many researchers. In 1941, D. H. Hyers [24] gave a first partial answer to Ulam's question and introduced the stability result as follows:

Theorem 1.7 [24] *Let E_1 and E_2 be two Banach spaces and $f: E_1 \to E_2$ be a function such that*

$$\| f(x+y) - f(x) - f(y) \| \leq \delta$$

for some $\delta > 0$ and for all $x,y \in E_1$. Then the limit

- *Muaadh Almahalebi, Department of Mathematics, Faculty of Sciences, Ibn Tofail University, BP 133 Kenitra, Morocco.*
- *E-mail: muaadh1979@hotmail.fr*
- *Sadeq AL-Ali, Department of Mathematics, Faculty of Sciences, Ibn Tofail University, BP 133 Kenitra, Morocco.*
- *E-mail: sadeqalali2018@gmail.com.*

$$A(x) := \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in E_1$, and $A: E_1 \to E_2$ is the unique additive function such that $\| f(x) - A(x) \| \le \delta$

for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

Later, T. Aoki [10] and D. G. Bourgin [11] considered the problem of stability with unbounded Cauchy differences. Th. M. Rassias [28] attempted to weaken the condition for the bound of the norm of Cauchy difference

$$\| f(x+y) - f(x) - f(y) \|$$

and proved a generalization of Theorem 1.7 using a direct method (cf. Theorem 1.8):

Theorem 1.8 *[28] Let E_1 and E_2 be two Banach spaces. If $f: E_1 \to E_2$ satisfies the inequality*

$$\| f(x+y) - f(x) - f(y) \| \le \theta(\| x \|^p + \| y \|^p)$$

for some $\theta \ge 0$, for some $p \in \mathbb{R}$ with $0 \le p < 1$, and for all $x, y \in E_1$, then there exists a unique additive function $A: E_{1_2}$ such that

$$\| f(x) - A(x) \| \le \frac{2\theta}{2 - 2^p} \| x \|^p$$

for each $x \in E_1$. If, in addition, $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

After then, Th. M. Rassias [29],[30] motivated Theorem 1.8 as follows:

Theorem 1.9 [29],[30] *Let E_1 be a normed space, E_2 be a Banach space, and $f: E_1 \to E_2$ be a function. If f satisfies the inequality*

$$\| f(x+y) - f(x) - f(y) \| \le \theta(\| x \|^p + \| y \|^p) \quad (1.1)$$

for some $\theta \ge 0$, for some $p \in \mathbb{R}$ with $p \ne 1$, and for all $x, y \in E_1 - \{0_{E_1}\}$, then there exists a unique additive function $A: E_1 \to E_2$ such that

$$\| f(x) - A(x) \| \le \frac{2\theta}{|2 - 2^p|} \| x \|^p \quad (1.2)$$

for each $x \in E_1 - \{0_{E_1}\}$.

Note that Theorem 1.9 reduces to Theorem 1.7 when $p = 0$. For $p = 1$, the analogous result is not valid. Also, J. Brzdek [12] showed that estimation (1.2) is optimal for $p \ge 0$ in the general case.

Recently, J. Brzdek [16] showed that Theorem 1.9 can be significantly improved; namely, in the case $p < 0$, each $f: E_1 \to E_2$ satisfying (1.1) must actually be additive, and the assumption of completeness of E_2 is not necessary. It is regrettable that this result does not remain valid if we restrict the domain of f (see the further detail in [18]). But then again, several mathematicians showed that the fixed point method is an another very efficient and convenient tool for proving the Hyers-Ulam stability for a quite wide class of functional equations (see [17]). Brzdek et al. [14] proved the fixed point theorem for a nonlinear operator in metric spaces and used this result to study the Hyers-Ulam stability of some functional equations in non-Archimedean metric spaces. In this work, they also obtained the fixed point result in arbitrary metric spaces as follows:

Theorem 1.10 [14] *Let X be a nonempty set, (Y, d) be a complete metric space, and $\Lambda: Y^X \to Y^X$ be a non-decreasing operator satisfying the hypothesis $\lim_{n\to\infty} \Lambda\delta_n = 0$*

for every sequence $\{\delta_n\}_{n\in\mathbb{N}}$ in Y^X with $\lim_{n\to\infty} \delta_n = 0$.

Suppose that $\mathcal{T}: Y^X \to Y^X$ is an operator satisfying the inequality

$$d(\mathcal{T}\xi(x), \mathcal{T}\mu(x)) \le \Lambda(\Delta(\xi, \mu))(x), \quad \xi, \mu \in Y^X,\ x \in X, \quad (1.3)$$

where $\Delta: Y^X \times Y^X \to \mathbb{R}_+^X$ is a mapping which is defined by

$$\Delta(\xi, \mu)(x) := d(\xi(x), \mu(x)) \quad \xi, \mu \in Y^X,\ x \in X. \quad (1.4)$$

If there exist functions $\varepsilon: X \to \mathbb{R}_+$ and $\varphi: X \to Y$ such that

$$d((\mathcal{T}\varphi)(x), \varphi(x)) \le \varepsilon(x) \quad (1.5)$$

and

$$\varepsilon^*(x) := \sum_{n\in\mathbb{N}_0} (\Lambda^n \varepsilon)(x) < \infty \quad (1.6)$$

for all $x \in X$, then the limit

$$\lim_{n\to\infty} ((\mathcal{T}^n \varphi))(x) \quad (1.7)$$

exists for each $x \in X$. Moreover, the function $\psi \in Y^X$ defined by

$$\psi(x) := \lim_{n\to\infty} ((\mathcal{T}^n \varphi))(x) \quad (1.8)$$

is a fixed point of $\mathcal{T}$ with

$$d(\varphi(x), \psi(x)) \le \varepsilon^*(x) \quad (1.9)$$

for all $x \in X$.

In 2013, Brzdek [15] gave the fixed point result by applying Theorem 1.10 as follows:

Theorem 1.11 [15] *Let X be a nonempty set, (Y, d) be a complete metric space, $f_1, \ldots, f_r: X \to X$ and $L_1, \ldots, L_r: X \to \mathbb{R}_+$ be given mappings. Suppose that $\mathcal{T}: Y^X \to Y^X$ and $\Lambda: \mathbb{R}_+^X \to \mathbb{R}_+^X$ are two operators satisfying the conditions*

$$d(\mathcal{T}\xi(x), \mathcal{T}\mu(x)) \le \sum_{i=1}^r L_i(x) d(\xi(f_i(x)), \mu(f_i(x))) \quad (1.10)$$

for all $\xi, \mu \in Y^X$, $x \in X$ and

$$\Lambda\delta(x) := \sum_{i=1}^r L_i(x) \delta(f_i(x)), \quad \delta \in \mathbb{R}_+^X,\ x \in X. \quad (1.11)$$

If there exist functions $\varepsilon: X \to \mathbb{R}_+$ and $\varphi: X \to Y$ such that

$$d(\mathcal{T}\varphi(x), \varphi(x)) \le \varepsilon(x) \quad (1.12)$$

and

$$\varepsilon^*(x) := \sum_{n=0}^\infty (\Lambda^n \varepsilon)(x) < \infty \quad (1.13)$$

for all $x \in X$, then the limit (1.7) exists for each $x \in X$. Moreover, the function (1.8) is a fixed point of $\mathcal{T}$ with (1.9) for all $x \in X$.

Then by using this theorem, Brzdek [15] improved, extended and complemented several earlier classical stability results concerning the additive Cauchy equation (in particular Theorem 1.9). Over the last few years, many mathematicians have investigated various generalizations, extensions and applications of the Hyers-Ulam stability of a number of functional equations (see, for instance, [1]-[5], [17], [18] and references therein); in particular, the stability problem of the radical functional equations in various spaces was proved in [7, 8, 9, 21, 20, 25, 26].

An analogue of Theorem 1.11 in 2-Banach spaces was stated and proved in [6].

Theorem 1.12 [6] *Let X be a nonempty set, $(Y, \|\cdot,\cdot\|)$ be a 2-Banach space, $g: X \to Y$ be a surjective mapping and let $f_1, \ldots, f_r: X \to X$ and $L_1, \ldots, L_r: X \to \mathbb{R}_+$ be given mappings. Suppose that $\mathcal{T}: Y^X \to Y^X$ and $\Lambda: \mathbb{R}_+^{X\times X} \to \mathbb{R}_+^{X\times X}$ are two operators satisfying the conditions*

$$\| \mathcal{T}\xi(x) - \mathcal{T}\mu(x), g(z) \| \le \sum_{i=1}^r L_i(x) \| \xi(f_i(x)) - \mu(f_i(x)), g(z) \| \quad (1.14)$$

for all $\xi, \mu \in Y^X$, $x, z \in X$ and

$$\Lambda\delta(x, z) := \sum_{i=1}^r L_i(x) \delta(f_i(x), z), \quad \delta \in \mathbb{R}_+^{X\times X},\ x, z \in X. \quad (1.15)$$

If there exist functions $\varepsilon: X \times X \to \mathbb{R}_+$ and $\varphi: X \to Y$ such that

$$\| \mathcal{T}\varphi(x) - \varphi(x), g(z) \| \le \varepsilon(x, z) \quad (1.16)$$

and

$$\varepsilon^*(x, z) := \sum_{n=0}^\infty (\Lambda^n \varepsilon)(x, z) < \infty \quad (1.17)$$

for all $x, z \in X$, then the limit

$$\lim_{n\to\infty} ((\mathcal{T}^n \varphi))(x) \quad (1.18)$$

exists for each $x \in X$. Moreover, the function $\psi: X \to Y$ defined by

$$\psi(x) := \lim_{n\to\infty} ((\mathcal{T}^n \varphi))(x) \quad (1.19)$$

is a fixed point of $\mathcal{T}$ with

$$\| \varphi(x) - \psi(x), g(z) \| \le \varepsilon^*(x, z) \quad (1.20)$$

for all $x, z \in X$.

In this paper, we achieve the general solutions of the following p-radical functional equation:

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=2f(x), \qquad (1.21)$$

where $p \geq 3$ is an odd natural number. In addition, we discuss the generalized Hyers-Ulam-Rassias stability problem and the hyperstability results in 2-Banach spaces by using Theorem 1.12 for the cosidered equation and the inhomogeneous p-radical functional equation related to Jensen mappings

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=2f(x)+G(x,y).$$

2 General solution of equation (1.21)

In this section, we give the general solution of functional equation (1.21). The proof of the following theorem has been patterned on the reasoning in [19].

Theorem 2.1 *Let Y be a linear space. A function $f:\mathbb{R}\to Y$ satisfies the functional equation (1.21) if and only if*

$$f(x)=F(x^p), \quad x\in\mathbb{R}, \qquad (2.1)$$

with some Jensen function $F:\mathbb{R}\to Y$.

Proof. Indeed, It is not hard to check without any problem that if $f:\mathbb{R}\to Y$ satisfies (2.1), then it is a solution to (1.21). On the other hand, if $f:\mathbb{R}\to Y$ is a solution of (1.21), then we write $F_0(x)=f(\sqrt[p]{x})$, for $x\in\mathbb{R}$. From (1.21) we obtain that

$$\begin{aligned}F_0(x+y)+F_0(x-y)&=f(\sqrt[p]{x+y})+f(\sqrt[p]{x-y})\\&=2f(\sqrt[p]{x})\\&=2F_0(x)\end{aligned}$$

for all $x,y\in\mathbb{R}$. It is enough to observe that there is a Jensen function $F:\mathbb{R}\to Y$ with $F(x)=F_0(x)$ for all $x\in\mathbb{R}$. This completes the proof.

3 Stability results of the p-radical functional equation (1.21)

In the following two theorems, we use Theorem 1.12 to investigate the generalized Hyers-Ulam stability of the functional equation (1.21) in 2-Banach spaces.

Hereafter, we assume that $(Y,\|\cdot,\cdot\|)$ is a 2-Banach space.

Theorem 3.1 *Let $h_1,h_2:\mathbb{R}^2\to\mathbb{R}_+$ be two functions such that*

$$\mathcal{U}:=\{n\in\mathbb{N}:\alpha_n:=2\lambda_1(n^p)\lambda_2(n^p)+\lambda_1(2n^p-1)\lambda_2(2n^p-1)<1\}\neq\phi, \qquad (3.1)$$

where

$$\lambda_i(n):=inf\{t\in\mathbb{R}_+:h_i(nx^p,z)\leq t\ h_i(x^p,z),\quad x,z\in\mathbb{R}\} \quad (3.2)$$

for all $n\in\mathbb{N}$, where $i=1,2$. Assume that $f:\mathbb{R}\to Y$ satisfies the inequality

$$\| f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-2f(x),g(z)\|\leq h_1(x^p,z)h_2(y^p,z) \qquad (3.3)$$

for all $x,y,z\in\mathbb{R}$ where $g:X\to Y$ be a surjective mapping. Then there exists a unique function $F:\mathbb{R}\to Y$ that satisfies the equation (1.21) such that

$$\| f(x)-F(x),g(z)\|\leq\lambda_0h_1(x^p,z)h_2(x^p,z) \qquad (3.4)$$

for all $x,z\in\mathbb{R}$, where

$$\lambda_0:=\inf_{n\in\mathcal{U}}\left\{\frac{\lambda_1(n^p)\lambda_2(2n^p-1)}{1-\alpha_n}\right\}.$$

Proof. Replacing x by mx and y by $\sqrt[p]{m^p-1}x$ where $x,y\in\mathbb{R}$ and $m\in\mathbb{N}$, in inequality (3.3), we get

$$\| f\left(\sqrt[p]{(2m^p-1)x^p}\right)-2f(mx)+f(x),g(z)\|\leq h_1(m^px^p,z)h_2((2m^p-1)x^p,z) \qquad (3.5)$$

for all $x,z\in\mathbb{R}$. For each $m\in\mathbb{N}$, we define the operator $\mathcal{T}_m:Y^{\mathbb{R}}\to Y^{\mathbb{R}}$ by

$$\mathcal{T}_m\xi(x):=2\xi(mx)-\xi\left(\sqrt[p]{(2m^p-1)x^p}\right), \quad \xi\in Y^{\mathbb{R}},x\in\mathbb{R}. \quad (3.6)$$

Further put

$$\varepsilon_m(x,z):=h_1(m^px^p,z)h_2((2m^p-1)x^p,z), \quad x,z\in\mathbb{R}, \quad (3.7)$$

and observe that

$$\begin{aligned}\varepsilon_m(x,z)&=h_1(m^px^p,z)h_2\left((2m^p-1)x^p,z\right)\\&\leq\lambda_1(m^p)\lambda_2(2m^p-1)h_1(x^p,z)h_2(x^p,z), \end{aligned}\qquad (3.8)$$

for all $x,z\in\mathbb{R}$ and all $m\in\mathbb{N}$. Then the inequality (3.5) takes the form

$$\| f(x)-\mathcal{T}_mf(x),g(z)\|\leq\varepsilon_m(x,z), \quad x,z\in\mathbb{R}. \qquad (3.9)$$

Furthermore, for every $x,z\in\mathbb{R}$, $\xi,\mu\in Y^{\mathbb{R}}$, we obtain

$$\begin{aligned}&\|\mathcal{T}_m\xi(x)-\mathcal{T}_m\mu(x),g(z)\|=\\&\| 2\xi(mx)-\xi\left(\sqrt[p]{(2m^p-1)x^p}\right)-2\mu(mx)+\mu\left(\sqrt[p]{(2m^p-1)x^p}\right),g(z)\|\\&\leq 2\|(\xi-\mu)(mx),g(z)\|+\|(\xi-\mu)\left(\sqrt[p]{(2m^p-1)x^p}\right),g(z)\|.\end{aligned}$$

This brings us to define the operator $\Lambda_m:\mathbb{R}_+^{\mathbb{R}\times\mathbb{R}}\to\mathbb{R}_+^{\mathbb{R}\times\mathbb{R}}$ by

$$\Lambda_m\delta(x,z):=2\delta(mx,z)+\delta\left(\sqrt[p]{(2m^p-1)x^p},z\right), \quad \forall\,\delta\in\mathbb{R}_+^{\mathbb{R}\times\mathbb{R}},x,z\in\mathbb{R}. \qquad (3.10)$$

For each $m\in\mathbb{N}$, the above operator has the form described in (1.15) with $f_1(x)=mx$, $f_2(x)=\left(\sqrt[p]{(2m^p-1)x^p}\right)$ and $L_1(x)=2$, $L_2(x)=1$ for all $x\in\mathbb{R}$. By induction, we will show that for each $x,z\in\mathbb{R}$, $n\in\mathbb{N}_0$, and $m\in\mathcal{U}$ we have

$$(\Lambda_m^n\varepsilon_m)(x,z)\leq\lambda_1(m^p)\lambda_2(2m^p-1)\alpha_m^nh_1(x^p,z)h_2(x^p,z) \qquad (3.11)$$

where

$$\alpha_m=2\lambda_1(m^P)\lambda_2(m^p)+\lambda_1(2m^p-1)\lambda_2(2m^p-1).$$

From (3.7) and (3.8), we obtain that the inequality (3.11) holds for $n=0$. Next, we will assume that (3.11) holds for $n=k$, where $k\in\mathbb{N}$. Then we have

$$\begin{aligned}(\Lambda_m^{k+1}\varepsilon_m)(x,z)&=\Lambda_m((\Lambda_m^k\varepsilon_m)(x,z))=\\&2(\Lambda_m^k\varepsilon_m)(mx,z)+(\Lambda_m^k\varepsilon_m)\left(\sqrt[p]{(2m^p-1)x^p},z\right)\\&\leq 2\lambda_1(m^p)\lambda_2(2m^p-1)\alpha_m^kh_1(m^px^p,z)h_2(m^px^p,z)\\&+\lambda_1(m^p)\lambda_2(2m^p-1)\alpha_m^kh_1((2m^p-1)x^p,z)h_2((2m^p-1)x^p,z)\\&\leq\lambda_1(m^p)\lambda_2(2m^p-1)\alpha_m^k(2\lambda_1(m^p)\lambda_2(m^p)\\&+\lambda_1(2m^p-1)\lambda_2(2m^p-1))h_1(x^p,z)h_2(x^p,z)\\&=\lambda_1(m^p)\lambda_2(2m^p-1)\alpha_m^{k+1}h_1(x^p,z)h_2(x^p,z)\end{aligned}$$

for all $x,z\in\mathbb{R}$, $m\in\mathcal{U}$. This shows that (3.11) holds for $n=k+1$. Now we can conclude that the inequality (3.11) holds for all $n\in\mathbb{N}_0$. Hence, we obtain

$$\begin{aligned}\varepsilon_m^*(x,z)&=\sum_{n=0}^{\infty}(\Lambda_m^n\varepsilon_m)(x,z)\\&\leq\sum_{n=0}^{\infty}\lambda_1(m^p)\lambda_2(2m^p-1)\alpha_m^nh_1(x^p,z)h_2(x^p,z)\\&=\frac{\lambda_1(m^p)\lambda_2(2m^p-1)h_1(x^p,z)h_2(x^p,z)}{1-\alpha_m}<\infty\end{aligned}$$

for all $x,z\in\mathbb{R}$, $m\in\mathcal{U}$. Therefore, according to Theorem 1.12 with $\varphi=f$ and $X=\mathbb{R}$ and using the surjectivity of g, we get that the limit $F_m(x):=\lim_{n\to\infty}(\mathcal{T}_m^nf)(x)$ exists for each $x\in\mathbb{R}$ and $m\in\mathcal{U}$, and

$$\| f(x)-F_m(x),g(z)\|\leq\frac{\lambda_1(m^p)\lambda_2(2m^p-1)}{1-\alpha_m}h_1(x^p,z)h_2(x^p,z), \quad x,z\in\mathbb{R},\ m\in\mathcal{U}. \qquad (3.12)$$

To prove that F_m satisfies the functional equation (1.21), just prove the following inequality

$$\| \mathcal{T}_m^n f\left(\sqrt[p]{x^p+y^p}\right)+\mathcal{T}_m^n f\left(\sqrt[p]{x^p-y^p}\right)-2\mathcal{T}_m^n f(x), g(z) \| \leq \alpha_m^n h_1(x^p,z)h_2(y^p,z) \quad (3.13)$$

for every $x,y,z \in \mathbb{R}$, $n \in \mathbb{N}_0$, and $m \in \mathcal{U}$. Since the case $n=0$ is just (3.3), take $k \in \mathbb{N}$ and assume that (3.13) holds for $n=k$ and every $x,y,z \in \mathbb{R}$, $m \in \mathcal{U}$. Then, for each $x,y,z \in \mathbb{R}$ and $m \in \mathcal{U}$, we get

$$\begin{aligned}
&\| \mathcal{T}_m^{k+1} f\left(\sqrt[p]{x^p+y^p}\right)+\mathcal{T}_m^{k+1} f\left(\sqrt[p]{x^p-y^p}\right)-2\mathcal{T}_m^{k+1} f(x), g(z) \| \\
&=\| 2\mathcal{T}_m^k f\left(m\sqrt[p]{x^p+y^p}\right)-\mathcal{T}_m^k f\left(\sqrt[p]{(2m^p-1)(x^p+y^p)}\right) \\
&+2\mathcal{T}_m^k f\left(m\sqrt[p]{x^p-y^p}\right)-\mathcal{T}_m^k f\left(\sqrt[p]{(2m^p-1)(x^p-y^p)}\right) \\
&-4\mathcal{T}_m^k f(mx)+2\mathcal{T}_m^k f\left(\sqrt[p]{(2m^p-1)x^p}\right), g(z) \| \leq \\
&2 \| \mathcal{T}_m^k f\left(m\sqrt[p]{x^p+y^p}\right)+\mathcal{T}_m^k f\left(m\sqrt[p]{x^p-y^p}\right)-2\mathcal{T}_m^k f(mx), g(z) \| \\
&+\| \mathcal{T}_m^k f\left(\sqrt[p]{(2m^p-1)(x^p+y^p)}\right)+\mathcal{T}_m^k f\left(\sqrt[p]{(2m^p-1)(x^p-y^p)}\right) \\
&\qquad -2\mathcal{T}_m^k f\left(\sqrt[p]{(2m^p-1)x^p}\right), g(z) \| \\
&\leq 2\alpha_m^k h_1(m^p x^p,z)h_2(m^p y^p,z) \\
&\qquad +\alpha_m^k h_1\left((2m^p-1)x^p,z\right)h_2\left((2m^p-1)y^p,z\right) \\
&\leq \alpha_m^k (2\lambda_1(m^p)\lambda_2(m^p) \\
&\qquad +\lambda_1(2m^p-1)\lambda_2(2m^p-1))h_1(x^p,z)h_2(y^p,z) \\
&\qquad = \alpha_m^{k+1} h_1(x^p,z)h_2(y^p,z).
\end{aligned}$$

Thus, by induction, we have shown that (3.13) holds for every $x,y,z \in \mathbb{R}$, $n \in \mathbb{N}_0$, and $m \in \mathcal{U}$. Letting $n \to \infty$ in (3.13), we obtain the equality

$$F_m\left(\sqrt[p]{x^p+y^p}\right)+F_m\left(\sqrt[p]{x^p-y^p}\right)=2F_m(x), \quad \forall\ x,y \in \mathbb{R}, m \in \mathcal{U}. \quad (3.14)$$

This implies that $F_m: \mathbb{R} \to Y$, defined in this way, is a solution of the equation

$$F(x)=2F(mx)+F\left(\sqrt[p]{(2m^p-1)x^p}\right), \quad x \in \mathbb{R}, m \in \mathcal{U}. \quad (3.15)$$

Next, we will prove that each p-radical function $F: \mathbb{R} \to Y$ satisfying the inequality

$$\| f(x)-F(x), g(z) \| \leq L\ h_1(x^p,z)h_2(x^p,z), \quad x,z \in \mathbb{R} \quad (3.16)$$

with some $L>0$, is equal to F_m for each $m \in \mathcal{U}$. To this end, we fix $m_0 \in \mathcal{U}$ and $F: \mathbb{R} \to Y$ satisfying (3.16). From (3.12), for each $x \in \mathbb{R}$, we get

$$\begin{aligned}
&\| F(x)-F_{m_0}(x), g(z) \| \leq \\
&\| F(x)-f(x), g(z) \| +\| f(x)-F_{m_0}, g(z) \| \\
&\leq L\ h_1(x^p,z)h_2(x^p,z)+\varepsilon_{m_0}^*(x,z) \\
&\leq L_0\ h_1(x^p,z)h_2(x^p,z)\textstyle\sum_{n=0}^{\infty} \alpha_{m_0}^n, \quad (3.17)
\end{aligned}$$

where $L_0 := (1-\alpha_{m_0})L+\lambda_1(m_0^p)\lambda_2(2m_0^p-1)>0$ and we exclude the case that $h_1(x^p,z) \equiv 0$ or $h_2(x^p,z) \equiv 0$ which is trivial. Observe that F and F_{m_0} are solutions to equation (3.15) for all $m \in \mathcal{U}$. Next, we show that, for each $j \in \mathbb{N}_0$, we have

$$\| F(x)-F_{m_0}(x), g(z) \| \leq L_0\ h_1(x^p,z)h_2(x^p,z)\sum_{n=j}^{\infty} \alpha_{m_0}^n, \quad \forall x,z \in \mathbb{R}. \quad (3.18)$$

The case $j=0$ is exactly (3.17). We fix $k \in \mathbb{N}$ and assume that (3.18) holds for $j=k$. Then, in view of (3.17), for each $x,z \in \mathbb{R}$, we get

$$\begin{aligned}
&\| F(x)-F_{m_0}(x), g(z) \| =\| 2F(m_0x)-F\left(\sqrt[p]{(2m_0^p-1)x^p}\right) \\
&-2F_{m_0}(m_0x)+F_{m_0}\left(\sqrt[p]{(2m_0^p-1)x^p}\right), g(z) \| \\
&\leq 2 \| F(m_0x)-F_{m_0}(m_0x), g(z) \| \\
&+\| F\left(\sqrt[p]{(2m_0^p-1)x^p}\right)-F_{m_0}\left(\sqrt[p]{(2m_0^p-1)x^p}\right), g(z) \| \\
&\leq 2L_0\ h_1(m_0^p x^p,z)h_2(m_0^p x^p,z)\textstyle\sum_{n=k}^{\infty} \alpha_{m_0}^n \\
&+L_0\ h_1((2m_0^p-1)x^p,z)h_2((2m_0^p-1)x^p,z)\sum_{n=k}^{\infty} \alpha_{m_0}^n \\
&= L_0\ (2h_1(m_0^p x^p,z)h_2(m_0^p x^p,z) \\
&+h_1((2m_0^p-1)x^p,z)h_2((2m_0^p-1)x^p,z))\sum_{n=k}^{\infty} \alpha_{m_0}^n \\
&\leq L_0\ \alpha_{m_0} h_1(x^p,z)h_2(x^p,z)\sum_{n=k}^{\infty} \alpha_{m_0}^n \\
&= L_0\ h_1(x^p,z)h_2(x^p,z)\sum_{n=k+1}^{\infty} \alpha_{m_0}^n.
\end{aligned}$$

This shows that (3.18) holds for $j=k+1$. Now we can conclude that the inequality (3.18) holds for all $j \in \mathbb{N}_0$. Now, letting $j \to \infty$ in (3.18), we get

$$F=F_{m_0}. \quad (3.19)$$

Thus, we have also proved that $F_m=F_{m_0}$ for each $m \in \mathcal{U}$, which (in view of (3.12)) yields

$$\| f(x)-F_{m_0}(x), g(z) \| \leq \frac{\lambda_1(m^p)\lambda_2(2m^p-1)}{1-\alpha_m} h_1(x^p,z)h_2(x^p,z), \quad x,z \in \mathbb{R},\ m \in \mathcal{U}. \quad (3.20)$$

This implies (3.4) with $F=F_{m_0}$ and (3.19) confirms the uniqueness of F.

The following theorem concerns the hyperstability of (1.21) in 2-Banach spaces. Namely, We consider functions $f: \mathbb{R} \to Y$ fulfilling (1.21) approximately, i.e., satisfying the inequality

$$\| f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-2f(x), g(z) \| \leq \eta(x,y,z), \quad \forall\ x,y,z \in \mathbb{R}, \quad (3.21)$$

with $\eta: \mathbb{R}^3 \to \mathbb{R}_+$ is a given mapping. Then we find a unique p-radical function $F: \mathbb{R} \to Y$ which is close to f. Then, under some additional assumptions on η, we prove that the conditional functional equation (1.21) is hyperstable in the class of functions $f: \mathbb{R} \to Y$, i.e., each $f: \mathbb{R} \to Y$ satisfying inequality (3.21), with such η, must fulfil equation (1.21).

Theorem 3.2 *Let h_1, h_2 and $\mathcal{U}$ be as in Theorem 3.1. Assume that*

$$\lim_{n\to\infty} \lambda_1(n)\lambda_2(n)=0 \quad (3.22)$$

Then every $f: \mathbb{R} \to Y$ satisfying (3.3) is a solution of (1.21).

Proof. Suppose that $f: \mathbb{R} \to Y$ satisfies (3.3). Then, by Theorem 3.1, there exists a mapping $F: \mathbb{R} \to Y$ satisfies (1.21) and

$$\| f(x)-F(x), g(z) \| \leq \lambda_0 h_1(x^p,z)h_2(x^p,z) \quad (3.23)$$

for all $x,z \in \mathbb{R}$, where

$$\lambda_0 := \inf_{n\in\mathcal{U}} \left\{\frac{\lambda_1(n^p)\lambda_2(2n^p-1)}{1-\alpha_n}\right\},$$

with

$$\alpha_n = 2\lambda_1(n^p)\lambda_2(n^p)+\lambda_1(2n^p-1)\lambda_2(2n^p-1).$$

Since, in view of (3.22), $\lambda_0=0$. This means that $f(x)=F(x)$ for all $x \in \mathbb{R}$, whence

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=2f(x), \quad x,y \in \mathbb{R}$$

which implies that f satisfies the functional equation (1.21) on $\mathbb{R}$.

4 Some Particular Cases

According to above theorems, we derive some particular cases from our main results.

Corollary 4.1 *Let $h_1, h_2 : \mathbb{R}^2 \to (0, \infty)$ be as in Theorem 3.1 such that*

$$\lim_{n\to\infty} \inf \sup_{x,z\in\mathbb{R}} \frac{h_1((2n^p-1)x^p,z)h_2((2n^p-1)x^p,z)+2h_1(n^px^p,z)h_2(n^px^p,z)}{h_1(x^p,z)h_2(x^p,z)} = 0.$$

Assume that $f:\mathbb{R}\to Y$ satisfies (1.21). Then there exist a unique p-radical function $F:\mathbb{R}\to Y$ and a unique constant $\kappa \in \mathbb{R}_+$ with

$$\| f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) - 2f(x), g(z) \| \leq \kappa\ h_1(x^p,z)h_2(x^p,z),$$

$\forall\ x,z \in \mathbb{R}$. (4.2)

Proof. By the definition of $\lambda_i(n)$ in Theorem 3.1, we observe that

$$2\lambda_1(n^p)\lambda_2(n^p) = 2\sup_{x,z\in\mathbb{R}} \frac{h_1(n^px^p,z)h_2(n^px^p,z)}{h_1(x^p,z)h_2(x^p,z)}$$
$$\leq 2\sup_{x,z\in\mathbb{R}} \frac{h_1((2n^p-1)x^p,z)h_2((2n^p-1)x^p,z)+2h_1(n^px^p,z)h_2(n^px^p,z)}{h_1(x^p,z)h_2(x^p,z)} \quad (4.3)$$

and

$$\lambda_1(2n^p-1)\lambda_2(2n^p-1) = \sup_{x,z\in\mathbb{R}} \frac{h_1((2n^p-1)x^p,z))h_2((2n^p-1)x^p,z))}{h_1(x^p,z)h_2(x^p,z)}$$
$$\leq \sup_{x,z\in\mathbb{R}} \frac{h_1((2n^p-1)x^p,z)h_2((2n^p-1)x^p,z)+h_1(n^px^p,z)h_2(n^px^p,z)}{h_1(x^p,z)h_2(x^p,z)} \quad (4.4)$$

Combining inequalities (4.3) and (4.4), we get

$$2\lambda_1(n^p)\lambda_2(n^p) + \lambda_1(2n^p-1)\lambda_2(2n^p-1) \leq 3\sup_{x,z\in\mathbb{R}} \frac{h_1((2n^p-1)x^p,z)h_2((2n^p-1)x^p,z)+2h_1(n^px^p,z)h_2(n^px^p,z)}{h_1(x^p,z)h_2(x^p,z)}. \quad (4.5)$$

Write

$$\gamma_n := \sup_{x,z\in\mathbb{R}} \frac{h_1((2n^p-1)x^p,z)h_2((2n^p-1)x^p,z)+2h_1(n^px^p,z)h_2(n^px^p,z)}{h_1(x^p,z)h_2(x^p,z)}.$$

From (4.1), there is a subsequence $\{\gamma_{n_k}\}$ of a sequence $\{\gamma_n\}$ such that $\lim_{k\to\infty}\gamma_{n_k} = 0$, that is,

$$\lim_{k\to\infty} \sup_{x,z\in\mathbb{R}} \frac{h_1((2n_k^p-1)x^p,z)h_2((2n_k^p-1)x^p,z)+2h_1(n_k^px^p,z)h_2(n_k^px^p,z)}{h_1(x^p,z)h_2(x^p,z)} = 0. \quad (4.6)$$

From (4.5) and (4.6), we find that

$$\lim_{k\to\infty} \lambda_1(2n_k^p-1)\lambda_2(2n_k^p-1) + 2\lambda_1(n_k^p)\lambda_2(n_k^p) = 0. \quad (4.7)$$

This implies

$$\lim_{k\to\infty} \frac{\lambda_1(n_k^p)\lambda_2(2n_k^p-1)}{1-\lambda_1(2n_k^p-1)\lambda_2(2n_k^p-1)-2\lambda_1(n_k^p)\lambda_2(n_k^p)} = \lim_{k\to\infty}\lambda_1(n_k^p)\lambda_2(2n_k^p-1) := \kappa$$

which means that λ_0 defined in Theorem 3.1 is equal to κ.

Corollary 4.2 *Let $\theta \geq 0$, $s,t,r \in \mathbb{R}$ such that $s+t<0$. Suppose that $f:\mathbb{R}\to Y$ satisfies the inequality*

$$\| f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) - 2f(x), g(z) \| \leq \theta|x^p|^s\ |y^p|^t\ |z|^r, \quad x,y,z \in \mathbb{R}\backslash\{0\}. \quad (4.8)$$

Then f satisfies (1.21) on $\mathbb{R}\backslash\{0\}$.

Proof. The proof follows from Theorem 3.1 by defining $h_1, h_2 : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}_+$ by $h_1(x^p,z) = \theta_1|x^p|^s|z|^{r_1}$ and $h_2(y^p,z) = \theta_2|y^p|^t|z|^{r_2}$, with $\theta_1,\theta_2 \in \mathbb{R}_+$ and $s,t,r_1,r_2 \in \mathbb{R}$ such that $\theta_1\theta_2 = \theta$, $r_1+r_2 = r$ and $s+t<0$.

For each $n \in \mathbb{N}$, we have

$$\lambda_1(n) = \inf\{t \in \mathbb{R}_+ : h_1(nx^p,z) \leq t\ h_1(x^p,z),\quad x,z \in \mathbb{R}\}$$
$$= \inf\{t \in \mathbb{R}_+ : \theta_1|\sqrt[p]{n}x|^{ps}|z|^{r_1} \leq t\ \theta_1|x|^{ps}|z|^{r_1},\quad x,z \in \mathbb{R}\backslash\{0\}\}$$
$$= n^s.$$

Also, we have $\lambda_2(n) = n^t$ for all $n \in \mathbb{N}$. Clearly, we can find $n_0 \in \mathbb{N}$ such that

$$\lambda_1(2n^p-1)\lambda_2(2n^p-1) + 2\lambda_1(n^p)\lambda_2(n^p) = (2n^p-1)^{s+t} + 2(n^p)^{s+t} < 1, \quad n \geq n_0. \quad (4.9)$$

According to Theorem 3.1, there exists a unique radical function $F:\mathbb{R}\backslash\{0\}\to Y$ such that

$$\| f(x) - F(x), g(z) \| \leq \theta\lambda_0|x|^{p(s+t)}|z|^r \quad (4.10)$$

for all $x,z \in \mathbb{R}\backslash\{0\}$, where

$$\lambda_0 := \inf_{n\geq n_0}\left\{\frac{\lambda_1(n^p)\lambda_2(2n^p-1)}{1-\lambda_1(2n^p-1)\lambda_2(2n^p-1)-2\lambda_1(n^p)\lambda_2(n^p)}\right\}.$$

On the other hand, Since $s+t<0$, one of s, t must be negative. Assume that $t<0$. Then

$$\lim_{n\to\infty}\lambda_1(n)\lambda_2(n) = \lim_{n\to\infty} n^{s+t} = 0. \quad (4.11)$$

Thus by Theorem 3.2, we get the desired results.

The next corollary prove the hyperstability results for the inhomogeneous p-radical functional equation.

Corollary 4.3 *Let $\theta,s,t,r \in \mathbb{R}$ such that $\theta \geq 0$ and $s+t<0$. Assume that $G:\mathbb{R}^2\to Y$ and $f:\mathbb{R}\to Y$ satisfy the inequality*

$$\| f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) - 2f(x) - G(x,y), g(z) \| \leq \theta|x^p|^s\ |y^p|^t\ |z|^r, \quad x,y,z \in \mathbb{R}\backslash\{0\}. \quad (4.12)$$

If the functional equation

$$f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) = 2f(x) + G(x,y),$$

$\forall\ x,y \in \mathbb{R}\backslash\{0\}$ (4.13)

has a solution $f_0:\mathbb{R}\to Y$, then f is a solution to (4.13).

Proof. From (4.12), we get that the function $K:\mathbb{R}\to Y$ defined by $K := f - f_0$ that satisfies (4.8). Consequently, Corollary 4.3 implies that K is a solution to the p-radical functional equation (1.21). Therefore,

$$f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) - 2f(x) - G(x,y) =$$
$$K\left(\sqrt[p]{x^p+y^p}\right) + f_0\left(\sqrt[p]{x^p+y^p}\right) + K\left(\sqrt[p]{x^p-y^p}\right) + f_0\left(\sqrt[p]{x^p-y^p}\right)$$
$$-2K(x) - 2f_0(x) - G(x,y)$$
$$= 0, \quad x,y \in \mathbb{R}\backslash\{0\}.$$

which means that f is a solution to (4.13).

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